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CONGRUENT DOMINATION NUMBER OF SOME CYCLE RELATED GRAPHS

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Abstract: A dominating set $D \subseteq V(G)$ is said to be a congruent dominating set(CDS) of G if

$$\sum_{v \in V(G)} d(v) \equiv 0 \left(\bmod \sum_{v \in D} d(v) \right)$$

The minimum cardinality of a minimal congruent dominating set of G is called the congruent domination number of G which is denoted by $\gamma_{cd}(G)$. We investigate congruent domination number of some cycle related graphs.

Keywords and Phrases: Dominating Set, Domination Number, Congruent Dominating Set, Congruent Domination Number.

2020 Mathematics Subject Classification: 05C07, 05C69.

1. Introduction and Preliminaries

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it due to its potential to solve the real life problems involving design and analysis of communication network as well as defence surveillance. Variety of domination models are available in the existing literature.

A brief account of dominating sets and its related concepts can be found in [2, 4, 5, 6, 7]. For standard notations and graph theoretic terminology, we will follow West [12] while the terms related to number theory are used in the sense of Burton [1].

We begin with finite, undirected and simple graph $G = (V(G), E(G))$ of order n. A set $D \subseteq V(G)$ of vertices in a graph G is called a dominating set if each vertex in $V(G) - D$ is adjacent to at least one vertex of D. A dominating set D is a minimal dominating set (MDS) if no proper subset D' of D is a dominating set of graph G. The domination number $\gamma(G)$ is the minimum cardinality of a minimal dominating set.

The open neighborhood $N(v)$ of a vertex $v \in V$ is the set of vertices adjacent to v, and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$.

The complement $\bar{G}$ of G is the graph with vertex set $V(G)$ and two vertices are adjacent in $\bar{G}$ if they are not adjacent in G.

The square graph G^2 of a graph G with vertex set $V(G)$ is the graph obtained by joining every pair of vertices which are at distance two in G.

Definition 1.1. *The duplication of a vertex v of a graph G produces a new graph G' by adding a vertex v' with $N(v') = N(v)$. In other words, a vertex v' is said to be duplication of v if all the vertices which are adjacent to v are now adjacent to v' also.*

Definition 1.2. *If the vertices of a graph G are duplicated altogether then the resultant graph is known as splitting graph of G, which is denoted as $S'(G)$.*

Definition 1.3. [3] *The double cover of a graph G with the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ is a bipartite graph G' with bipartition (X, Y) ; $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$, where two vertices x_i and y_j are adjacent if and only if v_i is adjacent to v_j in G.*

Definition 1.4. *The corona $G \bigodot H$ of two graphs G and H (with order n and m respectively) is defined as a graph obtained by taking one copy of G and n copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H.*

Definition 1.5. *The crown $C_n \bigodot K_1$ is a graph obtained by joining a pendant edge to each vertex of cycle C_n.*

Definition 1.6. *The armed crown is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge. It is denoted by ACr_n where n is the number of vertices in cycle C_n.*

The following new concept is recently introduced and explored by Vaidya and Vadhel [9, 10, 11].

A dominating set $D \subseteq V(G)$ is said to be a congruent dominating set (CDS) of G if

$$\sum_{v \in V(G)} d(v) \equiv 0 \left(\bmod \sum_{v \in D} d(v) \right) \tag{1}$$

A congruent dominating set $D \subseteq V(G)$ is said to be a minimal congruent dominating set if no proper subset D' of D is a congruent dominating set. The minimum cardinality of a minimal congruent dominating set of G is called the congruent domination number of G which is denoted by $\gamma_{cd}(G)$.

The following three results proved by Vaidya and Vadhel are stated here for ready reference.

Theorem 1.7. [9] *For any graph G, $1 \leqslant \gamma(G) \leqslant \gamma_{cd}(G) \leqslant n$.*

Theorem 1.8. [9] *For cycle C_n,*

$$\gamma_{cd}(C_n) = \begin{cases} \dfrac{n}{3} & ; \text{if } n \equiv 0 (\bmod 3) \\ \dfrac{n}{2} & ; \text{if } n \equiv 0 (\bmod 2) \text{ and } n \not\equiv 0 (\bmod 3) \\ n & ; \text{otherwise} \end{cases}$$

Theorem 1.9. [10] *Let G be an $r-$regular graph with $\gamma(G) = k$ then $\gamma_{cd}(G) = k+i$, where i is the minimum number of vertices in $V(G) - D$ such that ${}^{k+i}|_n$.*

Proposition 1.10. [8] *For $n \geqslant 3$,*

$$\gamma(S'(C_n)) = \begin{cases} \dfrac{n}{2} & ; \text{if } n \equiv 0 (\bmod 4) \\ \dfrac{n+1}{2} & ; \text{if } n \equiv 1, 3 (\bmod 4) \\ \dfrac{n+2}{2} & ; \text{if } n \equiv 2 (\bmod 4) \end{cases}$$

2. Main Results

Theorem 2.1. *For the complement of cycle C_n with $n > 3$, $\gamma_{cd}(\overline{C_n})$ = the least prime divisor of n.*

Proof. Let $V = \{v_1, v_2, ..., v_n\}$ be the set of vertices of $\overline{C_n}$.

Obviously, $d(v) = n - 3$; $\forall\, v \in V(G)$ and so

$$\sum_{v \in V(G)} d(v) = n(n-3) \tag{2}$$

Note that $\gamma(\overline{C_n}) = 2$ and so, $\gamma_{cd}(\overline{C_n}) \geqslant 2$.
Let $n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}$, for some $k \in \mathbb{N}$. Then each p_i divides $n(n-3)$; $i \in \{1, 2, ..., k\}$.
Now let $D = \{v_1, v_2, ..., v_m\} \subseteq V$ be a minimal CDS with minimum cardinality. Then $m \geqslant 2$ and degree sum of vertex set of the set D is $m(n-3)$.
Hence, $m(n-3)|n(n-3)$. That is, $m|n$.

Thus, $m|p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}$, for some $k \in \mathbb{N}$.

Since, D is a minimal CDS with minimum cardinality, we have,

$m = \min\{p_1, p_2, ..., p_k\}$.

Hence, $m =$ the least prime divisor of n and so, $\gamma_{cd}(\overline{C_n}) =$ the least prime divisor of n.

Illustration 2.2. *The complement of C_8 is shown in Figure 1 in which the set of solid vertices is a CDS with minimum cardinality. Hence, $\gamma_{cd}(\bar{C}_8) = 2$.*

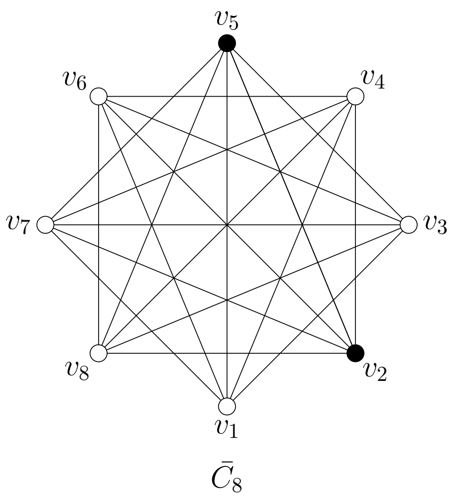

Figure 1: $\bar{C}_8$ and its CDS

Theorem 2.3. *For the square of the cycle C_n,*

$$\gamma_{cd}(C_n^2) = \begin{cases} 1 & ;\text{if n} = 3, 4, 5 \\ m & ;\text{if n} > 5 \end{cases}$$

where, m is a least positive integer such that $m \geqslant \left\lceil \frac{n}{5} \right\rceil$ and $n \equiv 0 (\text{mod } m)$.

Proof. Note that $C_n^2 = K_n$, for $n = 3, 4, 5$. Hence, $\gamma_{cd}(C_n^2) = 1$, for $n = 3, 4, 5$.

Now, we prove this theorem for $n > 5$.

Since C_n^2 is a 4−regular graph, from Theorem 1.9, we have $\gamma_{cd}(C_n^2) = \gamma(C_n^2) + i$, where i is the minimum number of vertices in $V(G) - D$ such that ${}^{\gamma(C_n^2)+i}|_n$ and D is a dominating set.
Let $m = \gamma(C_n^2) + i$. Then $m \geqslant \gamma(C_n^2)$ and so m is a least positive integer such that $m \geqslant \gamma(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$ and $m \equiv 0(\text{mod } n)$.
This theorem can be written in the following way also:

$$\gamma_{cd}(C_n^2) = \begin{cases} \frac{n}{5} & \text{; if n} \equiv 0(\text{mod } 5) \\ \frac{n}{4} & \text{; if n} \equiv 0(\text{mod } 4) \text{ \& n} \not\equiv 0(\text{mod } 5) \\ \frac{n}{3} & \text{; if n} \equiv 0(\text{mod } 3) \text{ \& n} \not\equiv 0(\text{mod } 4, 5) \\ \frac{n}{2} & \text{; if n} \equiv 0(\text{mod } 2) \text{ \& n} \not\equiv 0(\text{mod } 3, 4, 5) \\ n & \text{otherwise} \end{cases}$$

Illustration 2.4. *The square of C_{10} is shown in Figure 2 in which the set of solid vertices is a CDS with minimum cardinality. Hence, $\gamma_{cd}(C_{10}^2) = 2$.*

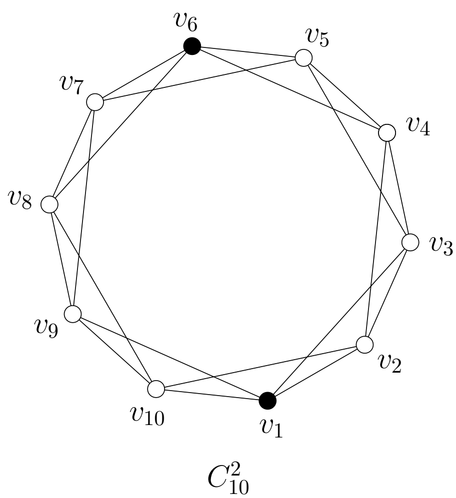

Figure 2: C_{10}^2 and its CDS.

Theorem 2.5. *For the splitting graph of cycle C_n,*

$$\gamma_{cd}(S'(C_n)) = 2\left\lceil \frac{n}{4} \right\rceil.$$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of C_n, which are duplicated by the vertices $v'_1, v'_2, ..., v'_n$ respectively. Then the resultant graph $S'(C_n)$ will have $2n$ vertices in which $v_1, v_2, ..., v_n$ are the vertices of degree 4 and $v'_1, v'_2, ..., v'_n$ are the vertices of degree 2.

Therefore,

$$\sum_{v \in V(S'(C_n))} d(v) = 6n \tag{3}$$

Case 1. If $n \equiv 0 (\bmod\ 4)$

Consider $D = \left\{v_{4k+1}, v_{4k+2}/0 \leqslant k \leqslant \frac{n}{4} - 1\right\}$ then $|D| = \frac{n}{2}$.
We claim that D is a minimal dominating set as for any $u \in D$, $D - \{u\}$ does not dominate at least one vertex of $N(u)$. Moreover, the degree sum of vertex set of a dominating set D is $2n$. Hence, D satisfies the condition (1) for being a CDS.

Now, by Theorem 1.7, we have $\gamma(S'(C_n)) \leqslant \gamma_{cd}(S'(C_n))$ and by Proposition 1.10, we have $\gamma(S'(C_n)) = \frac{n}{2}$ when $n \equiv 0 (\bmod\ 4)$. Consequently, $\gamma_{cd}(S'(C_n)) \geqslant \frac{n}{2}$ and we have $|D| = \frac{n}{2}$.

Thus, D is a minimal CDS with minimum cardinality.

Hence, $\gamma_{cd}(S'(C_n)) = \frac{n}{2} = 2\left\lceil \frac{n}{4} \right\rceil$.

Case 2. $n \equiv 1 (\bmod\ 4)$

For $n = 5$, consider $D = \{v_1, v'_6, v'_8, v'_9\}$ then $|D| = 4 = 2\left\lceil \frac{n}{4} \right\rceil$. Now,

$$\sum_{v \in D} d(v) = d(v_1) + d(v'_6) + d(v'_8) + d(v'_9) = 4 + 2 + 2 + 2 = 10 = 2n \tag{4}$$

For $n = 9$, consider $D = \{v_1, v_4, v_7, v'_1, v'_4, v'_7\}$ then $|D| = 6 = 2\left\lceil \frac{n}{4} \right\rceil$. Now,

$$\sum_{v \in D} d(v) = d(v_1) + d(v_4) + d(v_7) + d(v'_1) + d(v'_4) + d(v'_7) = 18 = 2n \tag{5}$$

For $n > 9$, consider $D = \left\{v_{4k+1}, v_{4k+2}, v_{n-5}, v'_{n-2}, v'_{n-3}, v'_{n-5}/0 \leqslant k \leqslant \frac{n-9}{4}\right\}$ then

$|D| = \dfrac{n+3}{2}$. Now,

$$\begin{aligned}\sum_{v \in D} d(v) &= \sum_{k=0}^{(n-9)/4} d(v_{4k+1}) + \sum_{k=0}^{(n-9)/4} d(v_{4k+2}) + d(v_{n-5}) + d(v'_{n-2}) + d(v'_{n-3}) + d(v'_{n-5}) \\ &= 4\left(\frac{n-9}{4}+1\right) + 4\left(\frac{n-9}{4}+1\right) + 4 + 2 + 2 + 2 \\ &= 2(n-5) + 10 \\ &= 2n \end{aligned} \tag{6}$$

Hence, D satisfies the condition (1) for being a CDS.

Moreover, D is a minimal dominating set as for any $u \in D$, $D - \{u\}$ does not dominate at least one vertex of $N[u]$.

Now, we will claim that D is of minimum cardinality.

By Proposition 1.10, $\gamma(S'(C_n)) = \dfrac{n+1}{2}$ when $n \equiv 1 (\text{mod } 4)$ and $|D| = \dfrac{n+3}{2}$.

Therefore, $|D| = \gamma(S'(C_n)) + 1$.

Thus, to prove that D is a minimal CDS with minimum cardinality it is enough to prove that there doesn't exist any CDS with cardinality $\dfrac{n+1}{2}$.

Note that there are n vertices of degree 4 as well as n vertices of degree 2 in which vertices of degree 2 are mutually non-adjacent. Looking to the nature of graph, it is essential to consider those vertex in a dominating set which are of degree 4 for the minimality. Let D' be such dominating set with $|D'| = \dfrac{n+1}{2}$ then degree sum of vertex set of D' is $2(n+1)$.

Now, by (3), (4), (5) and (6) we have, $6n \not\equiv 0(\text{mod } 2n+2)$ as $6n \equiv 0(\text{mod } 2n)$ and $n \equiv 1(\text{mod } 4)$. Therefore, D' is not a CDS.

Thus, D is a minimal CDS with minimum cardinality.

Hence, $\gamma_{cd}(S'(C_n)) = \dfrac{n+3}{2} = 2\left\lceil \dfrac{n}{4} \right\rceil$.

Case 3. $n \equiv 2(\text{mod } 4)$

For $n = 6$, consider $D = \{v_1, v_4, v'_1, v'_4\}$ then $|D| = 4 = 2\left\lceil \dfrac{n}{4} \right\rceil$. Now,

$$\sum_{v \in D} d(v) = d(v_1) + d(v_4) + d(v'_1) + d(v'_4) = 4 + 4 + 2 + 2 = 12 = 2n \tag{7}$$

For $n > 6$, consider $D = \left\{ v_{4k+1}, v_{4k+2}, v'_{n-2}, v'_{n-1} / 0 \leqslant k \leqslant \dfrac{n-6}{4} \right\}$ then $|D| =$

$\dfrac{n+2}{2}$. Now,

$$\begin{aligned}\sum_{v\in D} d(v) &= \sum_{k=0}^{(n-6)/4} d(v_{4k+1}) + \sum_{k=0}^{(n-6)/4} d(v_{4k+2}) + d(v'_{n-2}) + d(v'_{n-1})\\ &= 4\left(\frac{n-6}{4}+1\right) + 4\left(\frac{n-6}{4}+1\right) + 2 + 2\\ &= 2(n-2)+4\\ &= 2n \end{aligned} \tag{8}$$

Hence, D satisfies the condition (1) for being a CDS.

Now, we claim that D is a minimal dominating set as for any $u \in D$, $D - \{u\}$ will not dominate either u or at least one vertex of $N(u)$.
Now, for $n \equiv 2(\text{mod } 4)$, $\gamma(S'(C_n)) = \dfrac{n+2}{2}$, by Proposition 1.10 and we have $|D| = \dfrac{n+2}{2}$. So applying the same argument discussed in Case 1, we have, D is a minimal CDS with minimum cardinality.

Hence, $\gamma_{cd}(S'(C_n)) = \dfrac{n+2}{2} = 2\left\lceil \dfrac{n}{4} \right\rceil$.

Case 4. $n \equiv 3(\text{mod } 4)$

For $n = 3$, consider $D = \{v_1, v'_1\}$ then $|D| = 2 = 2\left\lceil \dfrac{n}{4} \right\rceil$. Now,

$$\sum_{v\in D} d(v) = d(v_1) + d(v'_1) = 4 + 2 = 6 = 2n \tag{9}$$

For $n > 3$, consider $D = \left\{ v_{4k+1}, v_{4k+2}, v_{n-2}, v'_{n-2} / 0 \leqslant k \leqslant \dfrac{n-7}{4} \right\}$ then $|D| = \dfrac{n+1}{2}$. Now,

$$\begin{aligned}\sum_{v\in D} d(v) &= \sum_{k=0}^{(n-7)/4} d(v_{4k+1}) + \sum_{k=0}^{(n-7)/4} d(v_{4k+2}) + d(v_{n-2}) + d(v'_{n-2})\\ &= 4\left(\frac{n-7}{4}+1\right) + 4\left(\frac{n-7}{4}+1\right) + 4 + 2\\ &= 2(n-3)+6\\ &= 2n \end{aligned} \tag{10}$$

Hence, D satisfies the condition (1) for being a CDS.

Now, we claim that D is a minimal dominating set as for any $u \in D$, $D - \{u\}$ will not dominate either u or at least one vertex of $N(u)$.
Now, by Proposition 1.10, $\gamma(S'(C_n)) = \dfrac{n+1}{2}$ when $n \equiv 3 (\mathrm{mod}\ 4)$ and we have $|D| = \dfrac{n+1}{2}$. So applying the same argument discussed in Case 1, we have, D is a minimal CDS with minimum cardinality.

Hence, $\gamma_{cd}(S'(C_n)) = \dfrac{n+1}{2} = 2\left\lceil \dfrac{n}{4} \right\rceil$.
We illustrate the above result.

Illustration 2.6. *Consider $S'(C_5)$ as shown in Figure 3, it is the case when $n \equiv 1 (\mathrm{mod}\ 4)$. The set of solid vertices is the CDS with minimum cardinality. Hence, $\gamma_{cd}(S'(C_5)) = 4$.*

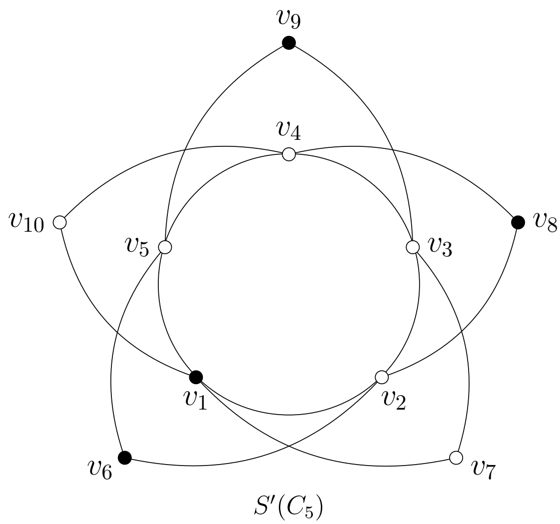

Figure 3: $S'(C_5)$ and its CDS.

Theorem 2.7. *Let G be the double cover of cycle C_n then*

$$\gamma_{cd}(G) = \begin{cases} \dfrac{2n}{3} & ;\text{if n} \equiv 0 (\mathrm{mod}\ 3) \\ n & ;\text{if n} \not\equiv 0 (\mathrm{mod}\ 3) \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n. Then,
$V(G) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$ is the set of vertices of graph G, where G is the

double cover of cycle C_n and $d(v) = 2,\ \forall v \in V(G)$. Moreover,

$$\sum_{v \in V(G)} d(v) = 4n \tag{11}$$

Case 1. n is even

In this case G is the disjoint union of two copies of cycle C_n.

Subcase 1. $n \equiv 0(\text{mod}3)$

Consider $D = \left\{v_{3k+1}, v'_{3k+1}/0 \leqslant k \leqslant \frac{n}{3} - 1\right\}$ then $|D| = \frac{2n}{3}$. Note that D is minimal dominating set with minimum cardinality as for $n \equiv 0(\text{mod } 3)$, $\gamma(C_n) = \frac{n}{3}$ and $\gamma(G) = 2\gamma(c_n) = \frac{2n}{3}$. Moreover, the degree sum of vertex set of a dominating set D is $2\left(\frac{2n}{3}\right) = \frac{4n}{3}$. Hence, D satisfies the condition (1) for being a CDS.
Since D is a minimal dominating set with minimum cardinality, it is also a minimal CDS with minimum cardinality.

Hence, $\gamma_{cd}(G) = \frac{2n}{3}$.

Subcase 2. $n \equiv 1(\text{mod}3)$

Consider $D = \{v_1, v_2, ..., v_n\}$ then $|D| = n$ and D is a minimal dominating set as for each $u \in D$, $D - \{u\}$ does not dominate vertex u. Moreover, the degree sum of vertex set of a dominating set D is $2n$. Hence, D satisfies the condition (1) for being a CDS.
Now, we will claim that D is a CDS with minimum cardinality.

Consider $n = 3k + 1$, for some $k \in \mathbb{N}$.
Note that $\gamma(G) = 2 \cdot \gamma(C_n) = 2\left(\frac{n+2}{3}\right)$ as for $n \equiv 1(\text{mod}3)$, $\gamma(C_n) = \frac{n+2}{3}$. Let S be such dominating set with $|S| = 2\left(\frac{n+2}{3}\right)$ then degree sum of the vertex set of S is $4\left(\frac{n+2}{3}\right) = 4(k+1)$.
Since S is the dominating set with minimum cardinality, $4(k + 1)$ is the least degree sum of vertices of the set S. Therefore, there does not exists any CDS S' with degree sum of vertex set being less than $4(k + 1)$.

Now,

$$\sum_{v \in V(G)} d(v) = 4n = 4(3k+1) \tag{12}$$

Then the set of divisors of the above degree sum is

$$\{l \in \mathbb{N}/\ l \mid (i \cdot (3k+1))\ ; i = 1, 2, 4\} \tag{13}$$

Moreover, $3k+1 < 4(k+1) < 2(3k+1)$; $k \in \mathbb{N} - \{1\}$.
We will claim that there doesn't exist any integer m with $3k+1 < m < 2(3k+1) \ni m \mid (4(3k+1))$.
If possible suppose $m \mid 4(3k+1)$, then $\exists\, d \in \mathbb{N} \ni 4(3k+1) = d \cdot m$

Now, $3k+1 < m < 2(3k+1)$
$\Rightarrow \dfrac{4(3k+1)}{3k+1} > d > \dfrac{4(3k+1)}{3k+1}$
$\Rightarrow 4 > d > 2$
$\Rightarrow d = 3$, which is contradiction as $3 \nmid (4(3k+1))$.

Hence from (12) and (13), it follows that the minimum possible degree sum of the CDS is $2(3k+1) = 2n$. This implies that D is a minimal CDS with minimum cardinality. Hence, $\gamma_{cd}(G) = n$.

Subcase 3. $n \equiv 2(\text{mod}3)$

Consider $D = \{v_1, v_2, ..., v_n\}$ then $|D| = n$ and D is a minimal dominating set as for each $u \in D$, $D - \{u\}$ does not dominate vertex u. Moreover, the degree sum of vertex set of a dominating set D is $2n$. Hence, D satisfies the condition (1) for being a CDS. Now, we will claim that D is a CDS with minimum cardinality.
Consider $n = 3k+2$, for some $k \in \mathbb{N}$.
Note that $\gamma(G) = 2 \cdot \gamma(C_n) = 2\left(\dfrac{n+1}{3}\right)$ as for $n \equiv 2(\text{mod}3)$, $\gamma(C_n) = \dfrac{n+1}{3}$. Let S be such dominating set with $|S| = 2\left(\dfrac{n+1}{3}\right)$ then degree sum of the vertex set of S is $4\left(\dfrac{n+1}{3}\right) = 4(k+1)$.
Since S is a dominating set with minimum cardinality, $4(k+1)$ is the least degree sum of vertices of the set S. Therefore, there does not exists any CDS S' with degree sum of vertex set being less than $4(k+1)$. Now,

$$\sum_{v \in V(G)} d(v) = 4n = 4(3k+2) \tag{14}$$

Then the set of divisors of the above degree sum is

$$\{l \in \mathbb{N} / \ l \mid (i \cdot (3k+2)) \ ; i = 1, 2, 4\} \tag{15}$$

Moreover, $3k+2 < 4(k+1) < 2(3k+2)$; $k \in \mathbb{N}$.

We will claim that there doesn't exist any integer m with $3k+2 < m < 2(3k+2) \ni m \mid (4(3k+2))$.
If possible suppose $m \mid 4(3k+2)$, then $\exists\, d \in \mathbb{N} \ni 4(3k+2) = d \cdot m$

Now, $3k+2 < m < 2(3k+2)$

$\Rightarrow \dfrac{4(3k+2)}{3k+2} > d > \dfrac{4(3k+2)}{3k+2}$
$\Rightarrow 4 > d > 2$
$\Rightarrow d = 3$, which is contradiction as $3 \nmid (4(3k+2))$.

Hence from (14) and (15), it follows that the minimum possible degree sum of the CDS is $2(3k+2) = 2n$.
This implies that D is a minimal CDS with minimum cardinality. Hence, $\gamma_{cd}(G) = n$.

Case 2. n is odd

In this case G is the cycle of order $2n$.

Subcase 1. $2n \equiv 0(\text{mod } 3)$

Then, $\gamma_{cd}(G) = \dfrac{2n}{3}$ as for $n \equiv 0(\text{mod}3)$, $\gamma(C_n) = \dfrac{n}{3}$.

Subcase 2. $2n \not\equiv 0(\text{mod } 3)$

Then, $\gamma_{cd}(G) = \dfrac{2n}{2} = n$ as for $n \equiv 0(\text{mod } 2)$, $\gamma(C_n) = \dfrac{n}{2}$. Hence,

$$\gamma_{cd}(G) = \begin{cases} \dfrac{2n}{3} & ;\text{if n} \equiv 0(\text{mod } 3) \\ n & ;\text{if } \text{ n} \not\equiv 0(\text{mod } 3) \end{cases}$$

Illustration 2.8. *In Figure 4, the cycle C_6 and its double cover G is shown in which the set of solid vertices is CDS with minimum cardinality. Hence, $\gamma_{cd}(G) = 4$.*

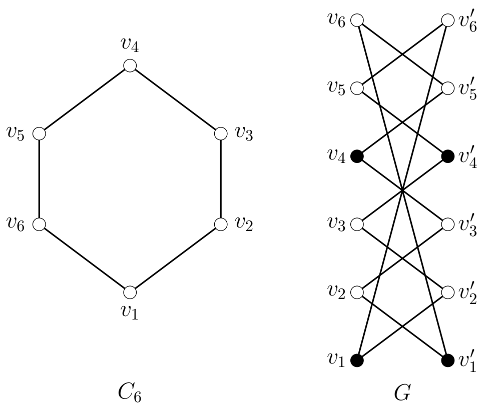

Figure 4: Cycle C_6 and its double cover G with its CDS.

Theorem 2.9. *For the crown $C_n \bigodot K_1$,*

$$\gamma_{cd}(C_n \bigodot K_1) = n.$$

Proof. Let $V(G) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n\}$ be the set of vertices of $G = C_n \bigodot K_1$ with $|V(G)| = 2n$. Here $v_1, v_2, ..., v_n$ are the vertices of degree 3 and $v'_1, v'_2, ..., v'_n$ are the vertices of degree 1. Therefore,

$$\sum_{v \in V(G)} d(v) = 4n \tag{16}$$

Consider $D = \{v'_1, v'_2, ..., v'_n\}$ then $|D| = n$. Now D is a minimal dominating set with minimum cardinality as $\gamma(C_n \bigodot K_1) = n$. Moreover, the degree sum of vertex set of a dominating set D is n. Hence, D satisfies the condition (1) for being a CDS.

Since D is a minimal dominating set with minimum cardinality, it is a minimal CDS with minimum cardinality.
Hence, $\gamma_{cd}(C_n \bigodot K_1) = n$.

Illustration 2.10. *In Figure 5, the crown graph is shown in which the set of solid vertices is a CDS with minimum cardinality. Hence, $\gamma_{cd}(C_7 \bigodot K_1) = 7$.*

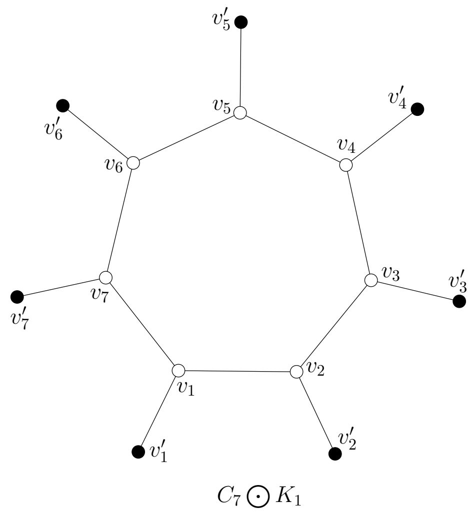

Figure 5: $C_7 \bigodot K_1$ and its CDS.

Theorem 2.11. *For the armed crown ACr_n,*

$$\gamma_{cd}(ACr_n) = n.$$

Proof. Let $V(G) = \{v_1, v_2, ..., v_n, v'_1, v'_2, ..., v'_n, v''_1, v''_2, ..., v''_n\}$ be the set of vertices of ACr_n with $|V(G)| = 3n$. Here, $v_1, v_2, ..., v_n$ are the vertices of degree 3, $v'_1, v'_2, ..., v'_n$ are the vertices of degree 2 and $v''_1, v''_2, ..., v''_n$ are the vertices of degree 1. Therefore,

$$\sum_{v \in V(G)} d(v) = 6n \tag{17}$$

Consider $D = \{v'_1, v'_2, ..., v'_n\}$ then $|D| = n$. Now D is a minimal dominating set with minimum cardinality as $\gamma(ACr_n) = n$. Moreover, the degree sum of vertex set of a dominating set D is $2n$. Hence, D satisfies the condition (1) for being a CDS. Since D is a minimal dominating set with minimum cardinality, it is a minimal CDS with minimum cardinality.

Hence, $\gamma_{cd}(ACr_n) = n$.

Illustration 2.12. *In Figure 6, the armed crown is shown in which the set of solid vertices is a CDS with minimum cardinality. Hence,* $\gamma_{cd}(ACr6) = 6$.

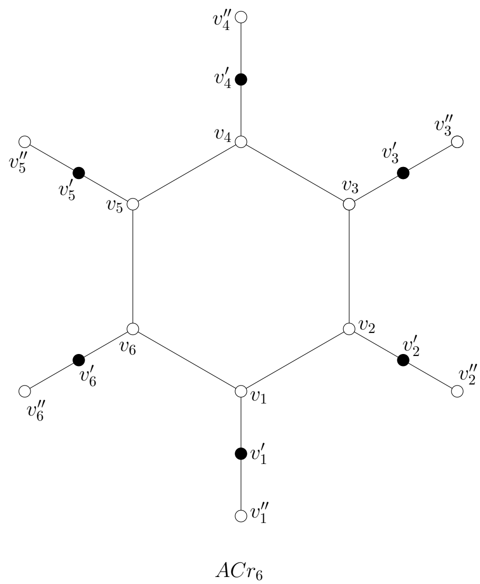

Figure 6: ACr_6 and its CDS.

3. Conclusion

The concept of congruent domination in graphs has been recently introduced by Vaidya and Vadhel [8]. The concept is a frontier between number theory and theory of graphs. The congruent domination numbers have been investigated for the graphs obtained from the cycle C_n.

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